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Electric and Magnetic Monopoles from a Lorentz-Covariant Hamiltonian

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In previous work we generalized the relation between the usual noncovariant Hamiltonian and the Poisson brackets to a covariant Hamiltonian and new brackets in the frame of Minkowski space. In the present paper we study the consequences of this new algebraic structure on the Lorentz Lie algebra defined in terms of these brackets. We show how a monopole with a dual electric–magnetic charge appears as a consequence of the conservation of the form of the standard Lorentz algebra symmetry. The breakdown of this symmetry is also envisaged.

1. INTRODUCTION

The derivation of the Maxwell equations from a Hamiltonian formalism is a well-known subject. One of the best ways to realize this goal is given, for example, in the book of Yourgrau and Mandelstam [1]. The Hamiltonian approach is very important because it opens the door to the generalization of other gauge theories like gravitation [2,3].

In a previous paper [4] we introduced, in a four-dimensional Minkowski space, a new kind of symplectic structure by means of brackets for studying the dynamics associated to a covariant Hamiltonian. These brackets define an algebraic structure between position- and velocity-dependent functions without an explicit formulation. We pointed out the link between these brackets and those used by Feynman in his derivation of the Maxwell equations [5–8]. The case of a curved space was also considered and the Christoffel symbols, covariant derivatives, and curvature tensors were expressed in terms of these brackets.

In the present paper we study the consequences of this new algebraic structure on the Lorentz Lie algebra defined in terms of these brackets. In

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ref. 8 considered the $sO(3)$ Lie algebraic structure in the context of the brackets used by Feynman, where the Dirac monopole appeared naturally as a consequence of the conservation of this Lie algebra. Here it will be shown that the Lorentz Lie algebra has a much richer structure. Actually the Lorentz symmetry induces a dual symmetry between a magnetic angular momentum and a new electric angular momentum leading to a general angular momentum equal to zero. We show how this duality can be broken.

2. COVARIANT HAMILTONIAN AND BRACKETS FORMALISM

In ref. 4 a generalization of the usual relation between noncovariant relativistic Hamiltonians and Poisson brackets to a Lorentz-covariant Hamiltonian H and new formal brackets in the frame of the Minkowski space was studied. Note that, in a different manner, Bracken also studied the relation between Feynman's problem and the Poisson brackets [9]. In this section we recall the fundamental relations of our algebraic structure.

The dynamic evolution law is given by means of a one real-parameter group of diffeomorphic transformations:

$$g(IR \times M_4) \to M_4: \quad g(\tau, x) = g^{\tau}x = x(\tau)$$

and the "velocity vector" associated to the particle is naturally introduced by

$$\dot{x}^{\mu} = \frac{d}{d\tau} g^{\tau} x^{\mu} \tag{1}$$

The derivative with respect to τ of an arbitrary function is defined on the tangent bundle space by the usual relation:

$$\frac{df(x, \dot{x}, \tau)}{d\tau} = [H, f(x, \dot{x}, \tau)] + \frac{\partial f(x, \dot{x}, \tau)}{\partial \tau} \tag{2}$$

where the Lorentz-covariant Hamiltonian H is defined as [4]

$$H = \frac{1}{2} m \frac{dx^{\mu}}{d\tau} \frac{dx_{\mu}}{d\tau} = \frac{1}{2} m\dot{x}^{\mu}\dot{x}_{\mu} \tag{3}$$

Equation (2) giving the dynamics of the system is the definition of our new brackets structure, and is the fundamental equation of this paper.

We require for these new brackets the usual first Leibniz law:

$$[A, BC] = [A, B]C + [A, C]B \tag{4}$$

and the skew symmetry

$$[A, B] = -[B, A] \tag{5}$$

where the quantities A, B, and C depend on x^μ and $\dot{x}^\mu$.

In the case of the vector position $x^\mu(\tau)$ we have from (2)

$$\dot{x}^\mu = [H, x^\mu] = m[\dot{x}^\nu, x^\mu]\dot{x}_\nu \tag{6}$$

and we easily deduce that

$$m[\dot{x}^\nu, x^\mu] = g^{\mu\nu} \tag{7}$$

where $g^{\mu\nu}$ is the metric tensor of the Minkowski space. Note that contrary to Feynman's work, the metric tensor is a consequence of our construction and is not imposed by hand, and is much more natural. In addition, the dynamics is given by the usual definition (2), whereas Feynman needed to add a supplementary Leibnitz law for the derivative with respect to time [4].

We impose the natural locality property

$$[x^\mu, x^\nu] = 0 \tag{8}$$

which directly gives for an expandable function of the position or the velocity the following useful relations:

$$[x^\mu, f(\dot{x})] = -\frac{1}{m}\frac{\partial f(\dot{x})}{\partial \dot{x}_\mu} \tag{9}$$

$$[\dot{x}^\mu, f(x)] = \frac{1}{m}\frac{\partial f(x)}{\partial x_\mu}$$

To compute the bracket between two components of the velocity we require the Jacobi identity:

$$[[\dot{x}^\mu, \dot{x}^\nu], x^\rho] + [[x^\rho, \dot{x}^\mu], \dot{x}^\nu] + [[\dot{x}^\nu, x^\rho]\, \dot{x}^\mu] = 0 \tag{10}$$

which leads by using (7) to the definition of the skew-symmetric "electromagnetic" tensor:

$$[\dot{x}^\mu, \dot{x}^\nu] = -\frac{qF^{\mu\nu}(x)}{m^2} \tag{11}$$

and the equation of motion is recovered as

$$\ddot{x}^\mu = \frac{d\dot{x}^\mu}{d\tau} = [H, \dot{x}^\mu] = \frac{q}{m}F^{\mu\nu}(x)\dot{x}_\nu \tag{12}$$

3. LORENTZ SYMMETRY

As an application of the formalism introduced in ref. 4 and recalled in the preceding section we study the consequence of this algebraic structure on the Lorentz symmetry. In ref. 8 we considered the $sO(3)$ Lie algebraic structure in the context of the brackets used by Feynman. Here it will be shown that the Lorentz Lie algebra has a richer structure. Actually the Lorentz symmetry induces a dual symmetry between a magnetic angular momentum and a new electric angular momentum leading to a general angular momentum equal to zero. We show how this duality can be broken.

Defining the four-angular momentum as

$$L^{\mu\nu} = m(x^{\mu}\dot{x}^{\nu} - x^{\nu}\dot{x}^{\mu}) \tag{13}$$

we find, by using the prior defined laws of the brackets, the standard Lorentz Lie algebra without electromagnetic field ($F^{\mu\nu} = 0$) and the transformation laws of the position and velocity by this symmetry:

$$\begin{aligned}
[x^{\mu}, L^{\rho\sigma}] &= g^{\mu\sigma}x^{\rho} - g^{\mu\rho}x^{\sigma} \\
[\dot{x}^{\mu}, L^{\rho\sigma}] &= g^{\mu\sigma}\dot{x}^{\rho} - g^{\mu\rho}\dot{x}^{\sigma} \\
[L^{\mu\nu}, L^{\rho\sigma}] &= g^{\mu\rho}L^{\nu\sigma} - g^{\nu\rho}L^{\mu\sigma} + g^{\mu\sigma}L^{\rho\nu} - g^{\nu\sigma}L^{\rho\mu}
\end{aligned} \tag{14}$$

These relations are the same as those obtained for the Poincaré Lie algebra with the four-momentum P^{μ}; the algebra in the tangent bundle space context is similar to this one built in the cotangent bundle space.

3.1. Case with a Gauge Curvature

We have seen that in a presence of a gauge curvature we have

$$[\dot{x}^{\mu}, \dot{x}^{\nu}] = -\frac{q}{m^2}F^{\mu\nu} \tag{15}$$

which leads to new algebraic relations:

$$\begin{aligned}
[x^{\mu}, L^{\rho\sigma}] &= g^{\mu\sigma}x^{\rho} - g^{\mu\rho}x^{\sigma} \\
[\dot{x}^{\mu}, L^{\rho\sigma}] &= g^{\mu\sigma}\dot{x}^{\rho} - g^{\mu\rho}\dot{x}^{\sigma} + \frac{g}{m}(F^{\mu\sigma}\dot{x}^{\rho} - F^{\mu\rho}\dot{x}^{\sigma}) \\
[L^{\mu\nu}, L^{\rho\sigma}] &= g^{\mu\rho}L^{\nu\sigma} - g^{\nu\rho}L^{\mu\sigma} + g^{\mu\sigma}L^{\rho\nu} - g^{\nu\sigma}L^{\rho\mu} \\
&\quad + q(x^{\mu}x^{\rho}F^{\nu\sigma} - x^{\nu}x^{\rho}F^{\mu\sigma} + x^{\mu}x^{\sigma}F^{\rho\nu} - x^{\nu}x^{\sigma}F^{\rho\mu})
\end{aligned} \tag{16}$$

Now we introduce a new generalized angular momentum:

$$\mathscr{L}^{\mu\nu} = L^{\mu\nu} + M^{\mu\nu} \tag{17}$$

where $M^{\mu\nu}$ are the components of a new electromagnetic angular momentum in order to keep the standard Lorentz Lie algebraic structure (14), that is,

$$[x^{\mu}, \pounds^{\rho\sigma}] = g^{\mu\sigma}x^{\rho} - g^{\mu\rho}x^{\sigma}$$
$$[\dot{x}^{\mu}, \pounds^{\rho\sigma}] = g^{\mu\sigma}\dot{x}^{\rho} - g^{\mu\rho}\dot{x}^{\sigma} \tag{18}$$
$$[\pounds^{\mu\nu}, \pounds^{\rho\sigma}] = g^{\mu\rho}\pounds^{\nu\sigma} - g^{\nu\sigma}\pounds^{\mu\sigma} + g^{\mu\sigma}\pounds^{\rho\nu} - g^{\nu\delta}\pounds^{\rho\mu}$$

leading to some conditions on $M^{\mu\nu}$: from the first relation we deduce that $M^{\mu\nu}$ is only position dependent and from the second one we have

$$[\dot{x}^{\mu}, M^{\rho\sigma}] = \frac{q}{m}(F^{\mu\sigma}x^{\rho} - F^{\mu\rho}x^{\sigma}) \tag{19}$$

Putting this result in the third equation, we get

$$g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} + g^{\mu\sigma}M^{\rho\nu} - g^{\nu\sigma}M^{\rho\mu}$$
$$= q(F^{\nu\sigma}x^{\mu}x^{\rho} - F^{\mu\sigma}x^{\nu}x^{\rho} + F^{\rho\mu}x^{\mu}x^{\sigma} - F^{\rho\mu}x^{\nu}x^{\sigma}) \tag{20}$$

Remark. The conservation of the four-angular momentum tensor is realized; indeed, the condition $[H, \pounds^{\mu\nu}] = 0$ is directly deduced from Eq. (18):

$$m[\dot{x}^{\mu}, \pounds^{\rho\sigma}]\dot{x}_{\mu} = m(g^{\mu\sigma}\dot{x}^{\rho} - g^{\mu\rho}\dot{x}^{\sigma})\dot{x}_{\mu} = 0$$

Let us consider the only interesting case, the projection of this relation on the three-dimensional Euclidean space, where $\mu = \rho = k$, $\nu = i$, and $\sigma = j$. Equation (20) becomes

$$M^{ij} = q(F^{ij}x^k x_k - F^j{}_k x^k x^i - F_k{}^i x^k x^j) \tag{21}$$

From

$$M^i = \varepsilon^i{}_{jk}M^{jk} \tag{22}$$

the same magnetic angular momentum as for the $sO(3)$ case is deduced, as expected [8]:

$$\vec{M} = -q(\vec{r}\vec{B})\,\vec{r} \tag{23}$$

where we have naturally defined the magnetic field by

$$B_i = \varepsilon_{ijk}F^{jk} \tag{24}$$

Putting down (23) into (21), we find

$$x^iB^j + x^jB^i = -x^jx^k\,\frac{\partial B_k}{\partial x_i} \tag{25}$$

which has a radial vector field centered at the origin solution:

$$\vec{B} = \frac{\vec{r}}{4\pi r^3} \tag{26}$$

For this field we obtain the Poincaré magnetic angular momentum [11]

$$\vec{M} = -\frac{qg}{4\pi}\frac{\vec{r}}{r} \tag{27}$$

The other cases are related to Lorentz boosts and have no simple interesting results. Projecting the Lorentz Lie algebra structure on the three-dimensional space, we recover the same angular momentum as for the $sO(3)$ symmetry defined by the brackets used by Feynman. To obtain new results, we consider now in addition the dual of the gauge field.

3.2. Case with Hodge Duality

We choose the following definition for the gauge curvature:

$$[x^\mu, \ddot{x}^\nu] = -[\dot{x}^\mu, \dot{x}^\nu] = \frac{1}{m^2}(qF^{\mu\nu} + g\,{*F^{\mu\nu}}) \tag{28}$$

where g is the magnetic charge of the magnetic monopole [12] and the *-operation is the Hodge duality. From this we deduce a generalization of the equation of motion [7]:

$$m\ddot{x} = qF^{\mu\nu}(x)\dot{x}_\upsilon + g\,{*F^{\mu\nu}}(x)\dot{x}_\upsilon \tag{29}$$

and in this situation the Lorentz Lie algebra structure becomes

$$\begin{aligned}
[x^\mu, L^{\rho\sigma}] &= g^{\mu\sigma}x^\rho - g^{\mu\rho}x^\sigma \\
[\dot{x}^\mu, L^{\rho\sigma}] &= g^{\mu\sigma}\dot{x}^\rho - g^{\mu\rho}\dot{x}^\sigma + \frac{g}{m}(F^{\mu\sigma}\dot{x}^\rho - F^{\mu\rho}\dot{x}^\sigma) \\
&\quad + \frac{g}{m}({*F^{\mu\sigma}}\dot{x}^\rho - {*F^{\mu\rho}}\dot{x}^\sigma) \qquad (30) \\
[L^{\mu\nu}, L^{\rho\sigma}] &= g^{\mu\rho}L^{\nu\sigma} - g^{\nu\rho}L^{\mu\sigma} + g^{\mu\sigma}L^{\rho\nu} - g^{\nu\sigma}L^{\rho\mu} \\
&\quad + q(x^\mu x^\rho F^{\nu\sigma} - x^\nu x^\rho F^{\mu\sigma} + x^\mu x^\sigma F^{\rho\nu} - x^\nu x^\sigma F^{\rho\mu}) \\
&\quad + g({*F^{\nu\sigma}}x^\mu x^\rho - {*F^{\mu\sigma}}x^\nu x^\rho + {*F^{\rho\nu}}x^\mu x^\sigma - {*F^{\rho\mu}}x^\nu x^\sigma)
\end{aligned}$$

We introduce a new generalized electromagnetic angular momentum:

$$\pounds^{\mu\nu} = L^{\mu\nu} + M^{\mu\nu} \tag{31}$$

where the constraint imposed on the tensor M by the Lie algebra structure (14) shows that it must be only position dependent and satisfies

$$[\dot{x}^\mu, M^{\rho\sigma}] = \frac{q}{m}(F^{\mu\sigma}x^\rho - F^{\mu\rho}x^\sigma) + \frac{g}{m}({*F^{\mu\sigma}}x^\rho - {*F^{\mu\rho}}x^\sigma) \tag{32}$$

Then Eq. (20) becomes

$$
\begin{aligned}
&g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} + g^{\mu\sigma}M^{\rho\mu} - g^{\nu\sigma}M^{\rho\mu} \\
&\quad = q(F^{\nu\sigma}x^{\mu}x^{\rho} - F^{\mu\sigma}x^{\nu}x^{\rho} + F^{\rho\nu}x^{\mu}x^{\sigma} - F^{\rho\mu}x^{\nu}x^{\sigma}) \\
&\qquad + g({}^{*}F^{\nu\sigma}x^{\mu}x^{\rho} - {}^{*}F^{\mu\sigma}x^{\nu}x^{\rho} + {}^{*}F^{\rho\mu}x^{\mu}x^{\sigma} - {}^{*}F^{\rho\mu}x^{\nu}x^{\sigma})
\end{aligned} \tag{33}
$$

and the result of the projection of this equation on the three-dimensional space is

$$
M^{ij} = q(F^{ij}x^k x_k - F^j{}_k x^k x^i - F_{k,}x^k x^j) + g({}^{*}F^{ij}x^k x_k - {}^{*}F^j{}_k x^k x^i - {}^{*}F_k{}^i x^k x^j) \tag{34}
$$

The new angular momentum is the sum of two contributions, a magnetic one (the same as before) and an electric one:

$$
\vec{M} = -q(\vec{r}\vec{B})\vec{r} + g(\vec{r}\vec{E})\vec{r} = \overrightarrow{M_m} + \overrightarrow{M_e} = -(\vec{r}\vec{P})\vec{r} \tag{35}
$$

where

$$
\begin{aligned}
\overrightarrow{M_m} &= -q(\vec{r}\vec{B})\vec{r} \\
\overrightarrow{M_e} &= g(\vec{r}\vec{E})\vec{r}
\end{aligned} \tag{36}
$$

are the magnetic and electric angular momenta, and

$$
\vec{P} = q\vec{B} - g\vec{E} \tag{37}
$$

Here the vector $\vec{P}$ plays the role of $\vec{B}$ in Eq. (23).

3.2.1 Solution with a Generalized Angular Momentum Equal to Zero

Reguiring now the Jacobi identity between the velocities

$$
[\dot{x}^{\mu}, [\dot{x}^{\nu}, \dot{x}^{\rho}]] + [\dot{x}^{\nu}, [\dot{x}^{\rho}, \dot{x}^{\mu}]] + [\dot{x}^{\rho}, [\dot{x}^{\mu}, \dot{x}^{\nu}]] = 0 \tag{38}
$$

we obtain the generalized Maxwell equations

$$
q(\partial^{\mu}F^{\nu\rho} + \partial^{\nu}F^{\rho\mu} + \partial^{\rho}F^{\mu\nu}) + g(\partial^{\mu *}F^{\nu\rho} + \partial^{\nu *}F^{\rho\mu} + \partial^{\rho *}F^{\mu\nu}) = 0 \tag{39}
$$

The projection of (39) on the three-dimensional space gives

$$
q \operatorname{div} \vec{B} - g \operatorname{div} \vec{E} = \operatorname{div}(\vec{P}) = 0 \tag{40}
$$

where $\vec{P}$ can be taken either perpendicular to the vector $\vec{r}$ or null. We then have

$$
\vec{M} = 0 \tag{41}
$$

The preceding Jacobi identity imposes either that there are no electric and magnetic monopoles or that the two monopoles exactly compensate each other. In the case without the dual field we did not impose this Jacobi identity in such a manner as to get a magnetic monopole and as a consequence the Poincaré angular momentum. By adding the dual field, the Jacobi identity does not forbid monopoles, but the the electric and magnetic contributions cancel the electromagnetic angular momentum. In the next section we relax this Jacobi identity and break the duality symmetry between the two monopoles.

3.2.2. *Solution with a Generalized Angular Momentum Different from Zero*

To break this duality symmetry, we introduce the tensor $N^{\mu\nu\rho}$ as

$$q(\partial^{\mu}F^{\nu\rho} + \partial^{\nu}F^{\rho\mu} + \partial^{\rho}F^{\mu\nu}) + g(\partial^{\mu *}F^{\nu\rho} + \partial^{\nu *}F^{\rho\mu} + \partial^{\rho *}F^{\mu\nu}) = qgN^{\mu\nu\rho} \tag{42}$$

If we still project on the three-dimensional space, we arrive at the result div $\vec{P} \neq 0$. Following the same reasoning as in the preceding section, we find the following equation for the field $\vec{P}$ [cf. (25)]:

$$x^iP^j + x^jP^i = -x^jx^k \frac{\partial P_k}{\partial x_i} \tag{43}$$

which has a radial vector field centered at the origin solution:

$$\vec{P} = \frac{\vec{r}}{4\pi r^3} \tag{44}$$

We get then a nonvanishing electromagnetic angular momentum:

$$\vec{M} = \overrightarrow{M_m} + \overrightarrow{M_e} = -\frac{qg}{4\pi}\frac{\vec{r}}{r} \tag{45}$$

and as the modulus of this momentum is constant along the r axis, the magnetic and electric charges are not independent; this is the famous Dirac quantization condition connecting these two charges. It appears clearly that the vector $\vec{P} = q\vec{B} - g\vec{E}$ plays the same role as the magnetic field in the case without dual field, and due to the fact that for these monopoles the source of the fields is localized at the origin,

$$\begin{aligned} \text{div } \vec{P} &= [\dot{x}^i, [\dot{x}^j, \dot{x}^k]] + [\dot{x}^j, [\dot{x}^k, \dot{x}^i]] + [\dot{x}^k, [\dot{x}^i, \dot{x}^j]] \\ &= q \text{ div } \vec{B} - g \text{ div } \vec{E} = \frac{qg}{4\pi}\left[r^l, \frac{r_l}{r^3}\right] = qg\, \delta^3(\vec{r}) \end{aligned} \tag{46}$$

for example, we can choose

$$\begin{aligned} \vec{B} &= \frac{g}{8\pi}\frac{\vec{r}}{r^3} \\ \vec{E} &= -\frac{g}{8\pi}\frac{\vec{r}}{r^3} \end{aligned} \tag{47}$$

We have found that $\vec{M}$ is a new angular momentum which is the sum of the Poincaré magnetic angular momentum plus an electric angular momentum; $\vec{B}$ is the field of a Dirac magnetic monopole and $\vec{E}$ is the electric field of an electric "Coulomb" monopole.

In addition, we remark that the generalized angular momentum

$$\vec{\pounds} = m(\vec{r} \wedge \dot{\vec{r}}) - (\vec{r} \cdot \vec{P})\, \vec{r} \tag{48}$$

is conserved:

$$\frac{d\vec{\pounds}}{dt} = m(\vec{r} \wedge \ddot{\vec{r}}) - \{\vec{r} \wedge (\dot{\vec{r}} \wedge \vec{P})\} = 0 \tag{49}$$

because the particle satisfies the usual equation of motion.

4. CONCLUSION

In this paper we studied the consequences of a new algebraic structure defined in the context of a Lorentz-covariant Hamiltonian introduced in ref. 4 on the Lorentz symmetry. A dual magnetic–electric monopole appears as a direct consequence of the conservation of the standard form of the Lorentz Lie algebra and could be connected to Schwinger's dyons [13]. The breakdown of one of the Jacobi identities plays a key role in this framework because it breaks the magnetic–electric duality and is responsible for the presence of a new generalized angular momentum.

This formalism can be a new approach for studying gauge theories. Two directions at least could be studied: a true application to the quantification problem, and the generalization to Dirac brackets. The significance of the τ parameter is still open; the connection with the canonical proper time introduced by Gill *et al.* [15, 16] is perhaps a good approach to examine this.

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